International Journal of Information and Computer Science
IJICS Volume 1, Issue 7, October 2012 PP. 178-187
ISSN(online) 2161-5381
ISSN(print) 2161-6450
www.iji-cs.org

Cubical Homology in Digital Images

Ismet Karaca[1] , Ozgur Ege[2]
[1, 2]Department of Mathematics, Ege University, Bornova, Izmir 35100, TURKEY
[1]ismet.karaca@ege.edu.tr, [2]ege-ozgur87@hotmail.com

(Abstract) In this article we study the digital cubical homology groups of digital images which are based on the cubical homology groups of topological spaces in algebraic topology. We investigate some fundamental properties of cubical homology groups of digital images. We also calculate cubical homology groups of certain 2-dimensional and 3-dimensional digital images. We give a relation between digital simplicial homology groups and digital cubical homology groups. Moreover we show that the Mayer-Vietoris Theorem need not be hold in digital images.

Keywords: Digital Topology; Digital Cubical Set; Digital Cubical Homology Group; Euler Characteristic; Mayer-Vietoris Theorem

1. INTRODUCTION

Topological invariants are useful in many applications related to digital imaging and geometric modeling, and homology is a classical one. The higher homotopy groups and homology groups are useful algebraic tools in a large number of topological problems, and are computational tools of algebraic topology. The digital fundamental group is a nice tool to classify the digital images with k-adjacency relations but it does not yield completely information in a great class of explicit problems. We need to establish a new algebraic scructure which is called the digital homology groups in order to classify the various digital images with k-adjacency relations. The digital cubical homology can be an important tool to classify digital images. Many researchers (Kaczynski, Mischaikow, Mrozek, Allili, Tannenbaum, Kalies, Watson, Pilarczyk, Zelazna and Kot) have studied cubical homology.

Kalies, Mischaikow and Watson [8] introduce a method for computing the homology groups of cellular complexes composed of cubes. The algorithm used in the homology computations is based on a local reduction procedure, and they give an estimate of its computational complexity. This estimate is rigorous in two dimensions, and they conjecture its validity in higher dimensions.

Allili, Mischaikow and Tannenbaum [1] combine a new method combinatorial topology to compute the number of connected components and holes of objects in a given image, and fast segmentation methods to extract the objects. Their computational method for determining the homology groups is based on a reduction process of the size of the chain complex by local simplification in such a way that the homology is preserved at each step.

Kaczynski, Mischaikow and Mrozek [7] present cubical sets and the algebra of cubical sets. Also, they define cubical homology and investigate its most elementary properties. They present a computational approach to homology with the hope that such a theory will prove beneficial to the analysis and understanding of todays complex geometric challenges. They give a relation between cubical complexes and image data. They compare with cubical and simplicial complexes.

Cubical complexes [7] have several nice properties that simplicial complexes do not share. Images and numerical computations naturally lead to cubical sets. Subdividing these cubes to obtain a triangulation is at this point artificial and increases the size of data significiantly. For example, it requires n! simplices to triangulate a single n-dimensional cube. Because cubical complexes are so rigid, they can be stored with a minimal amount of information. For example, an elementary cube can be described using one vertex. Specifying a simplex typically requires knowledge of all the vertices. A product of elementary cubes is an elementary cube, but a product of simplices is not a simplex. Cubes are more convenient than simplices for constructing products. It's easier to construct a cubic chain on a product $X \times Y$ given cubic chains on X and Y. Thus cubes are useful for local-to-global problems. Also cubes have a nice tensor product property and this is crucial for obtaining some homotopy classification results.

The main areas in which computational cubical homology are applications in digital image processing/analysis, dynamical systems and medicine/scructural biology. The main applications in medicine/scructural biology also concern digital imagery. For example, cubical homology has been used to extract topological information, e.g. cavities, directly from raw MRI/MRA-data. This usefulness of this is that creating an image is costly in terms of time and memory used by the computer and thus if we are only interested in topological information we only need to use cubical homology which is much cheaper and thus allows for more detailed calculations under the same conditions.

Kot [9] presents an algorithm based on the simplification of the boundaries and coboundaries of cubes in R^n. This

algorithm is an extension of the results presented in [8]. The algorithm obtained allows to compute the homology groups for two-dimensional cubical sets.

Mrozek, Pilarczyk and Zelazna [10] present a new reduction algorithm for the efficient computation of the homology of a cubical set. The algorithm is based on constructing a possible large acyclic subspace, and then computing the relative homology.

This paper is concerned with setting up more algebraic invariants for a digital image with k-adjacency. The purpose of this paper is to give complete algebraic presentation of cubical homology group of any objects in digital image. Some results related to cubical homology groups of a digital image are given. We also calculate the cubical homology group of certain 2-dimensional and 3-dimensional digital images. The paper is organized as follows: In Section 2 we introduce the general notions of digital images with k-adjacency relations. In Section 3 we define a cubical digital image, we introduce the algebra of digital cubical sets and we define the digital boundary operator. In Section 4 we define the cubical homology groups of digital images, we give some examples about cubical homology groups of digital images, we compute the digital cubical homology groups of a minimal simple curve that its digital simplicial homology groups are found by Boxer, Karaca and Oztel [4], we conclude that digital simplicial homology groups and digital cubical homology groups of a digital image need not be isomorphic, we define Euler characteristic of a digital cubical set and compute Euler characteristic of $MSS_6^{'}$. In Section 5 we show that the Mayer-Vietoris Theorem need not be hold in digital images.

2. PRELIMINARIES

Let Z be the set of integers. Then Z^n is the set of lattice points in the n-dimensional Euclidean space. A (binary) digital image is a finite subset of Z^n with an adjacency relation. A variety of adjacency relations are used in the study of digital images, including the following.

Definition 2.1 [3] For a positive integer l with $1 \le l \le n$ and distinct two points

$$p = (p_1, p_2, ..., p_n), q = (q_1, q_2, ..., q_n) \in Z^n \quad (2.1)$$

p and q are k_l - adjacent if

(1) there are at most l indices i such that $\left| p_i - q_i \right| = 1$ and

(2) for all other indices j such that $\left| p_j - q_j \right| \neq 1,\ p_j = q_j$.

Note that the notation k_l is the number of points $q \in Z^n$ that are adjacent to a given point $p \in Z^n$ in this sense. From Definition 2.1, we have the following;

(1) Two points p and q in Z^2 are 8-adjacent if they are distinct and differ by at most 1 in each coordinate.

(2) Two points p and q in Z^2 are 4-adjacent if they are 8-adjacent and differ in exactly one coordinate.

(3) Two points p and q in Z^3 are 26-adjacent if they are distinct and differ by at most 1 in each coordinate.

(4) Two points p and q in Z^3 are 18-adjacent if they are 26-adjacent and differ in at most two coordinate.

(5) Two points p and q in Z^3 are 6-adjacent if they are 18-adjacent and differ in exactly one coordinate.

(6) Let $k \in \{4,8,6,18,26\}$. A k - neighbor of a lattice point p is k -adjacent to p. More general adjacency relations are studied in [6].

Definition 2.2 [2] Let $a, b \in Z$ with $a < b$. A set of the form

$$[a,b]_Z = \{z \in Z \mid a \le z \le b\} \quad (2.2)$$

is called a digital interval.

Definition 2.2 [11] Let k be an adjacency relation defined on Z^n. A digital image $X \subset Z^n$ is k -connected if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, ..., x_r\}$ of points of a digital image X such that $x = x_0, y = x_r$ and x_i and x_{i+1} are k -neighbors where $i = 0,1,..., r-1$. A k -component of a digital image X is a maximal k -connected subset of X.

Let $X \in Z^{n_0}$ and $Y \in Z^{n_1}$ be digital images with k_0 -adjacency and k_1 -adjacency respectively. A function $f : X \to Y$ is said to be (k_0, k_1)-continuous [3] if for every k_0 -connected subset U of X, f(U) is a k_1 -connected subset of Y . Such a function is called digitally continuous.

Preposition 2.4 [3] Let $X \in Z^{n_0}$ and $Y \in Z^{n_1}$ be digital images with k_0 -adjacency and k_1 -adjacency respectively. Then the function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every k_0 -adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are a k_1 -adjacent in Y.

By a digital k -path from x to y in a digital image X, we mean a (2, k)-continuous function $f : [0,m]_Z \to X$ such that f(0) = x and f(m) = y. If f(0) = f(m) then the k -path is said to be closed. A simple closed k -path is considered as a sequence $\{f(0), f(1), ..., f(m-1)\}$ of images of the k -path $f : [0,m]_Z \to X$ such that f(i) and f(j) are k -adjacent if and only if j = i $\pm$ mod m.

Let $X \in Z^{n_0}$ and $Y \in Z^{n_1}$ be digital images with k_0-adjacency and k_1-adjacency respectively. A function $f : X \to Y$ is a (k_0, k_1)-homeomorphism (see [2]) if f is (k_0 , k_1)-continuous and bijective and further $f^{-1} : X \to Y$ is (k_1, k_0)-continuous. We use the notation $X \approx_{(k_0,k_1)} Y$.

3. DIGITAL CUBICAL SETS

We benefit from [7] to study cubical homology in digital images.

Definition 3.1 [7] An elementary digital interval is a digital interval $I \subset Z$ of the form

$$I = [a, a+1]_Z \text{ or } I = [a, a]_Z \tag{3.1}$$

for some a$\in$Z. If a digital interval contains only one point, then we write $[a]_Z = [a, a]_Z$ to simplify the notation.

Elementary digital intervals that consist of a single point are degenerate, whereas those of length 1 are nondegenerate. An elementary digital cube Q with 2n-adjacency is a finite product of elementary digital intervals, that is,

$$Q = I_1 \times I_2 \times \cdots I_n \subset Z^n \tag{3.2}$$

Where each I_i is an elementary digital interval. The number of nondegenerate components in Q is called the dimension of Q and it is denoted by dim Q. The set of all elementary digital cubes in Z^n is denoted by $K(Z^n)$ and those of dimension q by $K_q(Z^n)$.

A digital image (X, 2n)$\subset Z^n$ is cubical if (X, 2n) can be written as a finite union of elementary digital cubes. If (X, 2n)$\subset Z^n$ is a digital cubical set, then we use the following notation:

$$K(X,2n) = \{Q \in K : Q \subset (X,2n)\} \tag{3.3}$$

and

$$K_q(X,2n) = \{Q \in K(X,2n) : \dim Q = q\} \tag{3.4}$$

The elements of $K_0(X,k)$ are the vertices of (X,k) and the elements of $K_1(X,k)$ are the edges of (X,k). More generally, the elements of $K_q(X,k)$ are the digital q-cubes of (X,k).

Example 3.2 Consider the digital image

$X = [0,1]_Z \times [2,3]_Z \times [4,5]_Z \subset Z^3$ with 26-adjacency.

This is an elementary digital cube and hence, is a digital cubical set.

$K_2(X,26) = \{[0]\times[2,3]\times[4,5], [1]\times[2,3]\times[4,5],$
$[0,1]\times[2]\times[4,5], [0,1]\times[3]\times[4,5], [0,1]\times[2,3]\times[4], [0,1]\times[2,3]\times[5]\},$

$K_1(X,26) = \{[0]\times[2]\times[4,5], [0]\times[3]\times[4,5]$
$[0]\times[2,3]\times[4], [0]\times[2,3]\times[5], [1]\times[2]\times[4,5],$
$[1]\times[3]\times[4,5], [1]\times[2,3]\times[4], [1]\times[2,3]\times[5],$
$[0,1]\times[2]\times[4], [0,1]\times[2]\times[5], [0,1]\times[3]\times[4],$
$[0,1]\times[3]\times[5]\}$

$K_0(X,26) = \{[0]\times[2]\times[4], [0]\times[2]\times[5], [0]\times[3]\times[4],$
$[1]\times[2]\times[5], [0]\times[3]\times[5], [1]\times[2]\times[4], [1]\times[3]\times[4],$
$[1]\times[3]\times[5]\}.$

We now present an algebraic structure for digital cubical sets arbitrary dimension. With each elementary digital q-cube $Q \in K_q$ with κ-adjacency we associate $\hat{Q}$ called an elementary digital q-chain of Z^n with κ-adjacency. The set of all elementary digital q-chains of Z^n with κ-adjacency is denoted by

$$\hat{K}_q = \{\hat{Q} : Q \in K_q\} \tag{3.5}$$

and the set of all elementary digital chains of Z^n is given by

$$\hat{K} = \bigcup_{q=0}^{\infty} \hat{K}_q \tag{3.6}$$

Given any finite collection $\{\hat{Q}_1, \hat{Q}_2, \ldots \hat{Q}_n\} \subset K_q$ of q-dimensional elementary digital chains, we can consider sums of the form

$$c = \alpha_1 \hat{Q}_1 + \alpha_2 \hat{Q}_2 + \ldots + \alpha_n \hat{Q}_n \tag{3.7}$$

where α_i are arbitrary integers. If all the $\alpha_i = 0,$, then c = 0. The set of digital q-chains with κ-adjacency is denoted by C_q^κ. C_q^κ is an abelian group and it is a free abelian group with basis $\hat{K}_q$. For each $Q \in K_q$, define $\hat{Q} : K_q^n \to Z$ by

$$\hat{Q}(P) = \begin{cases} 1, P = Q \\ 0, P \neq Q \end{cases} \tag{3.8}$$

Definition 3.3 [7] The group C_q^κ of q-dimensional digital chains of Z^n with κ-adjacency is the free abelian group generated by the elementary digital chains of K_q. In particular, $\hat{K}_q$ is the basis for C_q^κ.

Note that there is a bijection between K_q and $\hat{K}_q$.

Definition 3.4 [7] Given two elementary digital cubes $P \in K_{q_1}$ and $Q \in K_{q_2}$, set

$$\hat{P} \Diamond \hat{Q} = P \times Q \tag{3.9}$$

This definition extends to arbitrary chains $c_1 \in C_{q_1}^{\kappa_1}$ and $c_2 \in C_{q_2}^{\kappa_2}$ by

$$c_1 \Diamond c_2 = \sum_{P \in K_{q_1}, Q \in K_{q_2}} \langle c_1, \hat{P} \rangle \langle c_2, \hat{Q} \rangle P \times Q \tag{3.10}$$

The chain $c_1 \Diamond c_2$ is called the digital cubical product of c_1 and c_2.

Definition 3.5 [7] Given $q \in Z,$ the cubical boundary operator

$$\partial_q : C_q^\kappa \to C_{q-1}^\kappa \tag{3.11}$$

is a homomorphism of free abelian groups, which is defined for an elementary chain $\hat{Q} \in \hat{K}_q$

$$\partial_q(\hat{Q}) = \begin{cases} 0, Q = [a]_Z \\ [a+1] - [a], Q = [a, a+1]_Z \end{cases} \tag{3.12}$$

Let $I = I_1(Q)$ and $P = I_2(Q) \times \ldots \times I_n(Q)$.Then $\hat{Q} = \hat{I} \Diamond \hat{P}$.Define

$$\partial_q(\hat{Q}) = \partial_{q_1} \hat{I} \Diamond \hat{P} + (-1)^{\dim I} \hat{I} \Diamond \partial_{q_2} \hat{P} \tag{3.13}$$

where q_1 = dim I and q_2 = dim P. Finally, we extend the definition to all chains by linearity; that is, if $c = \alpha_1 \hat{Q}_1 + \alpha_2 \hat{Q}_2 + \ldots + \alpha_m \hat{Q}_m$, then

$$\partial_q(c) = \alpha_1 \partial_q \hat{Q}_1 + \alpha_2 \partial_q \hat{Q}_2 + \ldots + \alpha_m \partial_q \hat{Q}_m \tag{3.14}$$

Proposition 3.6 [7] Let c_1^κ and c_2^κ be digital cubical chains with κ-adjacency; then

$$\partial_q(c_1^\kappa \Diamond c_2^\kappa) = \partial_q c_1^\kappa \Diamond c_2^\kappa + (-1)^{\dim c_1^\kappa} c_1^\kappa \Diamond \partial_q c_2^\kappa \tag{3.15}$$

Proposition 3.7 [7] $\partial_q \circ \partial_{q+1} = 0$ for all $q \geq 0$.

Definition 3.8 [7] The digital boundary operator for the digital cubical set (X, κ) is defined to be

$$\partial_q : C_q^\kappa(X) \to C_{q-1}^\kappa(X) \tag{3.16}$$

Let (X, κ) $\subset Z^n$ be a digital cubical set. Then $\partial_q(C_q^\kappa(X)) \subset C_{q-1}^\kappa(X)$.

4. CUBICAL HOMOLOGY IN DIGITAL IMAGES

Let X $\subset Z^n$ be a digital cubical set [7] with κ-adjacency. The kernel of

$$\partial_q : C_q^\kappa(X) \to C_{q-1}^\kappa(X) \tag{4.1}$$

is called the group of digitally cubical q-cycles in (X, κ) and denoted by $Z_q^\kappa(X)$. The image of

$$\partial_{q+1} : C_{q+1}^\kappa(X) \to C_q^\kappa(X) \tag{4.2}$$

is called the group of digitally cubical q-boundaries in (X, κ) and denoted by $B_q^\kappa(X)$.

Since Proposition 3.7, each digitally cubical q-boundary of digitally cubical (q+1)-chains is again a digitally cubical q-cycle, that is, $B_q^\kappa(X)$ is a normal subgroup of $Z_q^\kappa(X)$

for each $q \geq 0$. To give the nontrivial cycles, we introduce equivalence relation. We say that two cycles $z_1, z_2 \in Z_q^{\kappa}(X)$ are homologous and we write $z_1 \approx z_2$ if $z_1 - z_2 \in Z_q^{\kappa}(X)$ is a boundary in X, that is, $z_1 - z_2 \in B_q^{k}(X)$. The equivalence classes are elements of the quotient group $Z_q^{\kappa}(X) / B_q^{\kappa}(X)$.

Definition 4.1 [7] The qth digital cubical homology group is the quotient group

$$H_q^{\kappa}(X) = Z_q^{\kappa}(X) / B_q^{\kappa}(X).$$

The homology of X is the collection of all homology groups of X, that is,

$$H_*^{\kappa}(X) = \{H_q^{\kappa}(X)\}_{q \in Z} \qquad (4.3)$$

Given $z \in Z_q^{\kappa}(X)$, $[z] \in H_q^{\kappa}(X)$ is the homology class of z in X.

Example 4.2 [7] Let X $= \phi$. Then $C_q^{\kappa}(X) = 0$ for all q and hence

$$H_q^{\kappa}(X) = 0 \qquad q = 0,1,2,... \qquad (4.4)$$

Example 4.3 [7] Let X $= \{x_0\} \subset Z^n$ be a digital cubical set consisting of a single point with $\kappa -$adjacency. Then $x_0 = [a_1] \times [a_2] \times ... \times [a_n]$. Thus

$$C_q^{\kappa}(X) = \begin{cases} Z, k = 0 \\ 0, k \neq 0 \end{cases} \qquad (4.5)$$

Furthermore $Z_0^{\kappa}(X) \cong C_0^{\kappa}(X) = Z$. Since

$C_1^{\kappa}(X) = 0, B_0^{\kappa}(X) = 0$, and therefore, $H_0^{\kappa}(X) \cong Z$.

Since $C_q^{\kappa}(X) = 0$ for all $q \geq 1$, $H_q^{\kappa}(X) = 0$ for all $q \geq 1$. Therefore,

$$H_q^{\kappa}(X) = \begin{cases} Z, k = 0 \\ 0, k \neq 0. \end{cases} \qquad (4.6)$$

Example 4.4 [7] Let

$$X = [0]_Z \times [0,1]_Z \cup [1]_Z \times [0,1]_Z \cup [0,1]_Z \times [0]_Z \cup [0,1]_Z \times [1]_Z$$

be a digital cubical set with 4-adjacency. The sets of elementary digital cubes are

$$\kappa_0(X,4) = \{[0] \times [0], [0] \times [1], [1] \times [0], [1] \times [1]\}$$

$$\kappa_1(X,4) = \{[0] \times [0,1], [1] \times [0,1], [0,1] \times [0], [0,1] \times [1]\}$$

Thus the bases for the sets of chains are

$$\hat{K}_0(X,4) = \{[0]\hat{\times}[0],[0]\hat{\times}[1],[1]\hat{\times}[0],[1]\hat{\times}[1]\}$$

$$= \{[\hat{0}]\Diamond[\hat{0}],[\hat{0}]\Diamond[\hat{1}],[\hat{1}]\Diamond[\hat{0}],[\hat{1}]\Diamond[\hat{1}]\}$$

$$\hat{K}_1(X,4) = \{[0]\hat{\times}[0,1],[1]\hat{\times}[0,1],[0,1]\hat{\times}[0],[0,1]\hat{\times}[1]\}$$

$$= \{[\hat{0}]\Diamond[0\hat{,}1],[\hat{1}]\Diamond[0\hat{,}1],[0\hat{,}1]\Diamond[\hat{0}],[0\hat{,}1]\Diamond[\hat{1}]\}.$$

To simplification, in $\hat{\kappa}_0(X,4)$

$$\hat{a}_1 = [\hat{0}]\hat{\Diamond}[\hat{0}], \hat{a}_2 = [\hat{0}]\Diamond[\hat{1}], \hat{a}_3 = [\hat{1}]\Diamond[\hat{0}], \hat{a}_4 = [\hat{1}]\Diamond[\hat{1}],$$

in $\hat{\kappa}_1(X,4)$

$$\hat{b}_1 = [\hat{0}]\Diamond[0\hat{,}1], \hat{b}_2 = [\hat{1}]\Diamond[0\hat{,}1], \hat{b}_3 = [0\hat{,}1]\Diamond[\hat{0}], \hat{b}_4 = [0\hat{,}1]\Diamond[\hat{1}].$$

Then chain groups are

$C_0^4(X) = \{\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4\}$ and $C_1^4(X) = \{\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4\}$.

Thus, we get the following short sequence :

$$0 \xrightarrow{\partial_2} C_1^4(X) \xrightarrow{\partial_1} C_0^4(X) \xrightarrow{\partial_0} 0 \qquad (4.7)$$

To compute the boundary operator we need to compute the boundary of the basis elements.

$$\partial_1(\hat{b}_1) = -\hat{a}_1 + \hat{a}_2 \qquad \partial_1(\hat{b}_2) = -\hat{a}_3 + \hat{a}_4$$

$$\partial_1(\hat{b}_3) = -\hat{a}_1 + \hat{a}_3 \qquad \partial_1(\hat{b}_4) = -\hat{a}_2 + \hat{a}_4 \qquad (4.8)$$

To determine $Z_1^4(X)$, we need to know Ker ∂_1, that is, we need to solve the equation

$$\partial_1(\alpha_1\hat{b}_1 + \alpha_2\hat{b}_2 + \alpha_3\hat{b}_3 + \alpha_4\hat{b}_4) = \alpha_1\partial_1(\hat{b}_1) + \alpha_2\partial_1(\hat{b}_2) + \alpha_3\partial_1(\hat{b}_3) + \alpha_4\partial_1(\hat{b}_4)$$

$$= (-\alpha_1 - \alpha_3)(\hat{b}_1) + (\alpha_1 - \alpha_4)(\hat{b}_2) + (-\alpha_2 + \alpha_3)(\hat{b}_3) + (\alpha_2 + \alpha_4)(\hat{b}_4).$$

Solving the equation

$$(-\alpha_1-\alpha_3)(\hat{b}_1)+(\alpha_1-\alpha_4)(\hat{b}_2)+(-\alpha_2+\alpha_3)(\hat{b}_3)+(\alpha_2+\alpha_4)(\hat{b}_4)=0$$

we have

$$\alpha_1=-\alpha_2=-\alpha_3=\alpha_4.$$

Hence

$$Z_1^4(X)=\{\alpha(\hat{b}_1-\hat{b}_2-\hat{b}_3-\hat{b}_4)|\alpha\in Z\}\cong Z.$$

Since $C_2^4(X)=0, B_1^4(X)=0$ and hence

$$H_1^4(X)=Z_1^4(X)\cong Z.$$

We turn to computing $H_0^4(X)$. First observe that there is no solution to the equation

$$\partial_1(\alpha_1\hat{b}_1+\alpha_2\hat{b}_2+\alpha_3\hat{b}_3+\alpha_4\hat{b}_4)=\hat{b}_1.$$

This implies that $\hat{a}_1\notin B_0^4(X)$. On the other hand,

$$\partial_1(\hat{b}_1)=-\hat{a}_1+\hat{a}_2 \qquad \partial_1(\hat{b}_1+\hat{b}_4)=-\hat{a}_1+\hat{a}_4$$

$$\partial_1(\hat{b}_1+\hat{b}_4-\hat{b}_2)=-\hat{a}_1+\hat{a}_3.$$

Thus,

$$\{\hat{a}_1-\hat{a}_2,\hat{a}_1-\hat{a}_4,\hat{a}_1-\hat{a}_3\}\subset B_0^4(X).$$

In particular, all the elementary chains are homologous, that is,

$$\hat{a}_1\approx\hat{a}_2\approx\hat{a}_3\approx\hat{a}_4.$$

Now consider an arbitrary chain $z\in C_0^4(X)$.Then

$$z=\alpha_1\hat{a}_1+\alpha_2\hat{a}_2+\alpha_3\hat{a}_3+\alpha_4\hat{a}_4.$$

So on the level of homology

$$\begin{aligned}[z]_X &=[\alpha_1\hat{a}_1+\alpha_2\hat{a}_2+\alpha_3\hat{a}_3+\alpha_4\hat{a}_4]_X\\ &=\alpha_1[\hat{a}_1]_X+\alpha_2[\hat{a}_2]_X+\alpha_3[\hat{a}_3]_X+\alpha_4[\hat{a}_4]_X\\ &=(\alpha_1+\alpha_2+\alpha_3+\alpha_4)[\hat{a}_1]_X \qquad (4.9)\end{aligned}$$

where the last equality comes from that fact that all the elementary chains are homologous. Therefore, we can think of every element of $H_q^\kappa(X)=Z_q^\kappa(X)/B_q^\kappa(X)$ as being generated by $\hat{a}_1$ and thus $H_0^\kappa(X)\cong Z$.We have proven that

$$H_k^4(X)=\begin{cases}Z, k=0,1\\ 0, k\neq 0,1.\end{cases}$$

Theorem 4.5 If

$$MSS_6^{'}=\{c_0=(0,0,0),c_1=(1,0,0),c_2=(1,1,0),c_3=(0,1,0),$$

$$c_4=(0,0,1),c_5=(1,0,1),c_6=(1,1,1),c_7=(0,1,1)\},$$

That is,

$$MSS_6^{'}=[0,1]_Z\times[0,1]_Z\times[0,1]_Z\subset Z^3$$

is a digital cubical set with 6-adjacency (see figure 1), then its digital cubical homology groups are

$$H_q^6(MSS_6^{'})=\begin{cases}Z, q=0,2\\ 0, q\neq 0,2.\end{cases}$$

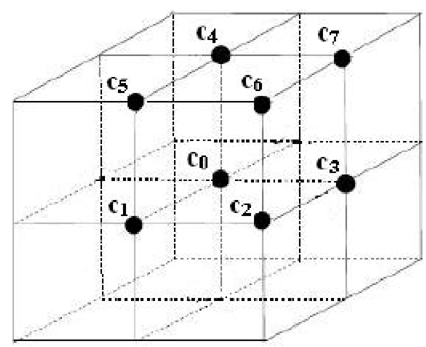

Figure 1: [5] $MSS_6^{'}$

Proof. The sets of elementary digital cubes of $MSS_6^{'}$ are

$$\kappa_2(MSS_6^{'},6)=\{[0,1]\times[0,1]\times[0],[0,1]\times[0]\times[0,1],[0,1]\times[1]\times[0,1]$$

$$[1]\times[0,1]\times[0,1],[0]\times[0,1]\times[0,1],[0,1]\times[0,1]\times[1]\}$$

$$\kappa_1(MSS_6^{'},6)=\{[0,1]\times[0]\times[0],[0]\times[0]\times[0,1],[0]\times[0,1]\times[0],$$

$$[1]\times[0,1]\times[0],[1]\times[0]\times[0,1],[1]\times[1]\times[0,1],[0,1]\times[1]\times[0],$$

$$[0]\times[1]\times[0,1],[0,1]\times[0]\times[1],[0]\times[0,1]\times[1],[1]\times[0,1]\times[1],$$

$[0,1]\times[1]\times[1]\}$

$\kappa_0(MSS'_6,6)=\{[0]\times[0]\times[0],[1]\times[0]\times[0],[1]\times[1]\times[0],$

$[0]\times[1]\times[0],[0]\times[0]\times[1],[1]\times[0]\times[1],[1]\times[1]\times[1],$

$[0]\times[1]\times[1]\}.$

Thus the bases for the sets of chains are

$$\widehat{\mathcal{K}_2}(MSS'_6,6)=\{\widehat{[0,1]}\diamond\widehat{[0,1]}\diamond\widehat{[0]},\widehat{[0,1]}\diamond\widehat{[0]}\diamond\widehat{[0,1]},\widehat{[0,1]}\diamond\widehat{[1]}\diamond\widehat{[0,1]},$$
$$\widehat{[1]}\diamond\widehat{[0,1]}\diamond\widehat{[0,1]},\widehat{[0]}\diamond\widehat{[0,1]}\diamond\widehat{[0,1]},\widehat{[0,1]}\diamond\widehat{[0,1]}\diamond\widehat{[1]}\}$$

$$\widehat{\mathcal{K}_1}(MSS'_6,6)=\{\widehat{[0,1]}\diamond\widehat{[0]}\diamond\widehat{[0]},\widehat{[0]}\diamond\widehat{[0]}\diamond\widehat{[0,1]},\widehat{[0]}\diamond\widehat{[0,1]}\diamond\widehat{[0]},$$
$$\widehat{[1]}\diamond\widehat{[0,1]}\diamond\widehat{[0]},\widehat{[1]}\diamond\widehat{[0]}\diamond\widehat{[0,1]},\widehat{[1]}\diamond\widehat{[1]}\diamond\widehat{[0,1]},$$
$$\widehat{[0,1]}\diamond\widehat{[1]}\diamond\widehat{[0]},\widehat{[0]}\diamond\widehat{[1]}\diamond\widehat{[0,1]},\widehat{[0,1]}\diamond\widehat{[0]}\diamond\widehat{[1]},$$
$$\widehat{[0]}\diamond\widehat{[0,1]}\diamond\widehat{[1]},\widehat{[1]}\diamond\widehat{[0,1]}\diamond\widehat{[1]},\widehat{[0,1]}\diamond\widehat{[1]}\diamond\widehat{[1]}\}$$

$$\widehat{\mathcal{K}_0}(MSS'_6,6)=\{\widehat{[0]}\diamond\widehat{[0]}\diamond\widehat{[0]},\widehat{[1]}\diamond\widehat{[0]}\diamond\widehat{[0]},\widehat{[1]}\diamond\widehat{[1]}\diamond\widehat{[0]},\widehat{[0]}\diamond\widehat{[1]}\diamond\widehat{[0]},$$
$$\widehat{[0]}\diamond\widehat{[0]}\diamond\widehat{[1]},\widehat{[1]}\diamond\widehat{[0]}\diamond\widehat{[1]},\widehat{[1]}\diamond\widehat{[1]}\diamond\widehat{[1]},\widehat{[0]}\diamond\widehat{[1]}\diamond\widehat{[1]}\}.$$

To simplify the notation, in $\hat{\kappa}_2(MSS'_6,6)$

$$\hat{e}_0=[0\hat{,}1]\Diamond[0\hat{,}1]\Diamond[\hat{0}],\hat{e}_1=[0\hat{,}1]\Diamond[\hat{0}]\Diamond[0\hat{,}1],$$
$$\hat{e}_2=[0\hat{,}1]\Diamond[\hat{1}]\Diamond[0\hat{,}1],\hat{e}_3=[\hat{1}]\Diamond[0\hat{,}1]\Diamond[0\hat{,}1],$$
$$\hat{e}_4=[\hat{0}]\Diamond[0\hat{,}1]\Diamond[0\hat{,}1],\hat{e}_5=[0\hat{,}1]\Diamond[0\hat{,}1]\Diamond[\hat{1}],$$

in $\hat{\kappa}_1(MSS'_6,6)$

$$\hat{d}_0=[0\hat{,}1]\Diamond[\hat{0}]\Diamond[\hat{0}],\hat{d}_1=[\hat{0}]\Diamond[\hat{0}]\Diamond[0\hat{,}1],$$
$$\hat{d}_2=[\hat{0}]\Diamond[0\hat{,}1]\Diamond[\hat{0}],\hat{d}_3=[\hat{1}]\Diamond[0\hat{,}1]\Diamond[\hat{0}],,$$
$$\hat{d}_4=[\hat{1}]\Diamond[\hat{0}]\Diamond[0\hat{,}1],\hat{d}_5=[\hat{1}]\Diamond[\hat{1}]\Diamond[0\hat{,}1],$$
$$\hat{d}_6=[0\hat{,}1]\Diamond[\hat{1}]\Diamond[\hat{0}],\hat{d}_7=[\hat{0}]\Diamond[\hat{1}]\Diamond[0\hat{,}1],$$
$$\hat{d}_8=[0\hat{,}1]\Diamond[\hat{0}]\Diamond[\hat{1}],\hat{d}_9=[\hat{0}]\Diamond[0\hat{,}1]\Diamond[\hat{1}],$$
$$\hat{d}_{10}=[\hat{1}]\Diamond[0\hat{,}1]\Diamond[\hat{1}],\hat{d}_{11}=[0\hat{,}1]\Diamond[\hat{1}]\Diamond[\hat{1}],$$

in $\hat{\kappa}_0(MSS'_6,6)$

$$\hat{c}_0=[\hat{0}]\Diamond[\hat{0}]\Diamond[\hat{0}],\hat{c}_1=[\hat{1}]\Diamond[\hat{0}]\Diamond[\hat{0}],$$
$$\hat{c}_2=[\hat{1}]\Diamond[\hat{1}]\Diamond[\hat{0}],\hat{c}_3=[\hat{0}]\Diamond[\hat{1}]\Diamond[\hat{0}],$$
$$\hat{c}_4=[\hat{0}]\Diamond[\hat{0}]\Diamond[\hat{1}],\hat{c}_5=[\hat{1}]\Diamond[\hat{0}]\Diamond[\hat{1}],$$
$$\hat{c}_6=[\hat{1}]\Diamond[\hat{1}]\Diamond[\hat{1}],\hat{c}_7=[\hat{0}]\Diamond[\hat{1}]\Diamond[\hat{1}].$$

Then chain groups are

$$C_2^6(MSS'_6)=\{\hat{e}_0,\hat{e}_1,\hat{e}_2,\hat{e}_3,\hat{e}_4,\hat{e}_5\}$$

$$C_1^6(MSS'_6)=\{\hat{d}_0,\hat{d}_1,\hat{d}_2,\hat{d}_3,\hat{d}_4,\hat{d}_5,\hat{d}_6,\hat{d}_7,\hat{d}_8,\hat{d}_9,\hat{d}_{10},\hat{d}_{11}\}$$

$$C_0^6(MSS'_6)=\{\hat{c}_0,\hat{c}_1,\hat{c}_2,\hat{c}_3,\hat{c}_4,\hat{c}_5,\hat{c}_6,\hat{c}_7\}.$$

Thus, we get the following short sequence :

$$0\xrightarrow{\partial_3}C_2^6(MSS'_6)\xrightarrow{\partial_2}C_1^6(MSS'_6)\xrightarrow{\partial_1}$$
$$C_0^6(MSS'_6)\xrightarrow{\partial_0}0 \qquad (4.10)$$

To compute the boundary operator we need to compute the boundary of the basis elements.

$$\partial_2(\hat{e}_0)=\hat{d}_3+\hat{d}_6-\hat{d}_4-\hat{d}_0$$

$$\partial_2(\hat{e}_1)=\hat{d}_0+\hat{d}_1-\hat{d}_8-\hat{d}_4$$

$$\partial_2(\hat{e}_2)=\hat{d}_6+\hat{d}_7-\hat{d}_{11}-\hat{d}_5$$

$$\partial_2(\hat{e}_3)=\hat{d}_3+\hat{d}_5-\hat{d}_{10}-\hat{d}_4$$

$$\partial_2(\hat{e}_4)=\hat{d}_2+\hat{d}_7-\hat{d}_9-\hat{d}_1$$

$$\partial_2(\hat{e}_5)=\hat{d}_{10}+\hat{d}_{11}-\hat{d}_9-\hat{d}_8.$$

To determine $Z_2^6(MSS'_6)$, we need to know Ker ∂_2, that is, we need to solve the equation

$$\partial_2(a_0\hat{e}_0+a_1\hat{e}_1+a_2\hat{e}_2+a_3\hat{e}_3++a_4\hat{e}_4+a_5\hat{e}_5)=$$
$$(-a_0+a_1)\hat{d}_0+(a_1-a_4)\hat{d}_1+(-a_0+a_4)\hat{d}_2+(a_0+a_3)\hat{d}_3$$
$$+(-a_1-a_3)\hat{d}_4+(-a_2+a_3)\hat{d}_5+(a_0+a_2)\hat{d}_6$$
$$+(a_2+a_4)\hat{d}_7+(-a_1-a_5)\hat{d}_8$$
$$+(-a_4-a_5)\hat{d}_9+(-a_3+a_5)\hat{d}_{10}+(-a_2+a_5)\hat{d}_{11}.$$

Solving the equation

$$(-a_0+a_1)\hat{d}_0+(a_1-a_4)\hat{d}_1+(-a_0+a_4)\hat{d}_2+(a_0+a_3)\hat{d}_3$$
$$+(-a_1-a_3)\hat{d}_4+(-a_2+a_3)\hat{d}_5+(a_0+a_2)\hat{d}_6$$
$$+(a_2+a_4)\hat{d}_7+(-a_1-a_5)\hat{d}_8$$
$$+(-a_4-a_5)\hat{d}_9+(-a_3+a_5)\hat{d}_{10}+(-a_2+a_5)\hat{d}_{11}=0$$

we have

$$a_0=a_1=-a_2=-a_3=a_4=-a_5.$$

Hence we get

$$Z_2^6(MSS_6^{'})=\{a(\hat{e}_0+\hat{e}_1-\hat{e}_2-\hat{e}_3+\hat{e}_4-\hat{e}_5:a_i\in Z\}\cong Z.$$

Since $B_2^6(MSS_6^{'})=\{0\}$, it follows that $H_2^6(MSS_6^{'})\cong Z$.

To compute the boundary operator we need to compute the boundary of the basis elements.

$$\partial_1(\hat{d}_0)=\hat{c}_4-\hat{c}_0 \qquad \partial_1(\hat{d}_7)=\hat{c}_7-\hat{c}_3$$
$$\hat{\partial}_1(\hat{d}_2)=\hat{c}_3-\hat{c}_0 \qquad \partial_1(\hat{d}_3)=\hat{c}_7-\hat{c}_4$$
$$\partial_1(\hat{d}_4)=\hat{c}_5-\hat{c}_1 \qquad \partial_1(\hat{d}_5)=\hat{c}_6-\hat{c}_2$$
$$\partial_1(\hat{d}_6)=\hat{c}_2-\hat{c}_1 \qquad \partial_1(\hat{d}_7)=\hat{c}_6-\hat{c}_5$$
$$\partial_1(\hat{d}_8)=\hat{c}_1-\hat{c}_0 \qquad \partial_1(\hat{d}_9)=\hat{c}_5-\hat{c}_4$$
$$\partial_1(\hat{d}_{10})=\hat{c}_2-\hat{c}_3 \qquad \partial_1(\hat{d}_{11})=\hat{c}_6-\hat{c}_7$$

To determine $Z_1^6(MSS_6^{'})$, we need to know Ker ∂_1, that is, we need to solve the equation

$$\partial_1(a_0\hat{d}_0+a_1\hat{d}_1+a_2\hat{d}_2+a_3\hat{d}_3++a_4\hat{d}_4+a_5\hat{d}_5+$$
$$a_6\hat{d}_6+a_7\hat{d}_7+a_8\hat{d}_8+a_9\hat{d}_9+a_{10}\hat{d}_{10}+a_{11}\hat{d}_{11})=$$
$$(-a_0-a_1-a_2)\hat{c}_0+(a_0-a_3-a_4)\hat{c}_1+(a_3-a_5+a_6)\hat{c}_2$$
$$+(a_2-a_6-a_7)\hat{c}_3+(a_1-a_8-a_{10})\hat{c}_4+(a_4-a_9+a_{11})\hat{c}_5$$
$$+(a_5+a_9+a_{10})\hat{c}_6+(a_7+a_8-a_{10})\hat{c}_7 \qquad (4.11)$$

Solving the equation

$$(-a_0-a_1-a_2)\hat{c}_0+(a_0-a_3-a_4)\hat{c}_1+(a_3-a_5+a_6)\hat{c}_2$$
$$+(a_2-a_6-a_7)\hat{c}_3+(a_1-a_8-a_{10})\hat{c}_4$$
$$+(a_4-a_9+a_{11})\hat{c}_5+(a_5+a_9+a_{10})\hat{c}_6$$
$$+(a_7+a_8-a_{10})\hat{c}_7=0$$

we have

$$a_2=-a_0-a_1 \qquad a_4=a_0-a_3 \qquad a_6=-a_3+a_5$$
$$a_7=-a_0-a_1+a_3-a_5 \qquad a_9=a_1-a_8$$
$$a_{10}=-a_0-a_1+a_3-a_5+a_8$$
$$a_{11}=-a_5-a_{10}=-a_0+a_3-a_5-a_8.$$

Hence we get

$$Z_1^6(MSS_6^{'})=\{a_0\hat{d}_0+a_1\hat{d}_1+(-a_0-a_1)\hat{d}_2+a_3\hat{d}_3+$$
$$(a_0-a_3)\hat{d}_4+a_5\hat{d}_5+(-a_3+a_5)\hat{d}_6+(-a_0-a_1+a_3-a_5)\hat{d}_7$$
$$+a_8\hat{d}_8+(a_1-a_8)\hat{d}_9+(-a_0-a_1+a_3-a_5+a_8)\hat{d}_{10}+$$
$$(-a_0+a_3-a_5-a_8)\hat{d}_{11}:a_i\in Z\}\cong Z^5.$$

We now compute $B_1^6(MSS_6^{'})$. From the equation (4.11),

we have

$$B_1^6(MSS_6^{'})=\{(k_0+k_1)\hat{d}_0+k_0\hat{d}_1+k_1\hat{d}_2+$$
$$(-k_1-k_3-k_4)\hat{d}_3+(-k_0+k_3+k_4)\hat{d}_4+k_2\hat{d}_5+$$
$$(-k_1-k_2-k_3-k_4)\hat{d}_6+(k_2+k_3+k_4)\hat{d}_7+$$
$$(-k_0+k_3)\hat{d}_8+k_3\hat{d}_9+k_4\hat{d}_{10}+(k_2+k_4)\hat{d}_{11}$$
$$:k_i\in Z, i=0,1,2,3,4\}\cong Z^5.$$

Since $B_1^6(MSS_6^{'})=Z_1^6(MSS_6^{'})$, it follows that

$$H_1^6(MSS_6^{'})\cong\{0\}.$$

We now compute $H_0^6(MSS_6^{'})$.

$$Z_0^6(MSS_6^{'})=\{a_0\hat{c}_0+a_1\hat{c}_1+a_2\hat{c}_2+a_3\hat{c}_3+a_4\hat{c}_4+a_5\hat{c}_5+$$
$$a_6\hat{c}_6+a_7\hat{c}_7:a_i\in Z\}\cong Z^8.$$

Any 0-cycle

$$u_0=a_0\hat{c}_0+a_1\hat{c}_1+a_2\hat{c}_2+a_3\hat{c}_3+a_4\hat{c}_4+a_5\hat{c}_5+$$
$$a_6\hat{c}_6+a_7\hat{c}_7$$

can be written as

$$u_0 = \partial_1((a_1 + a_2 + a_5 + a_6)\hat{d}_0 + a_2\hat{d}_1 + (a_3 + a_7)\hat{d}_3 + a_4\hat{d}_4 + (a_5 + a_6)\hat{d}_5 + a_6\hat{d}_6 + a_7\hat{d}_7) + \sum_{i=0}^{7} a_i\hat{c}_0.$$

Thus, u_0 is homologous to the 0-chain

$$(a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7)\hat{c}_0.$$

Therefore, $H_0^6(MSS_6^{'}) \cong Z.$ Thus we have the required result:

$$H_q^6(MSS_6^{'}) = \begin{cases} Z, q = 0,2 \\ 0, q \neq 0,2. \end{cases}$$

□

The digital simplicial homology groups of $MSS_6^{'}$ are found that

$$H_q^6(MSS_6^{'}) = \begin{cases} Z, q = 0 \\ Z^5, q = 1 \\ 0, q \neq 0,1 \end{cases}$$

(see [4]). Then we can state the following.

Proposition 4.6 The cubical and simplicial homology groups of a digital image need not be isomorphic.

We now define Euler characteristic for digital cubical sets.

Definition 4.7 Let (X,k) be a digital cubical set.

$$\beta_p = rankH_p^k(X) \tag{4.12}$$

is called the pth Betti number of $(X,k).$ The Euler characteristic of a digital cubical set (X,k) is the alternating sum of its Betti numbers (see [7]),

$$\chi(X,k) = \sum_{p=0}^{\infty}(-1)^p \beta_p.$$

Example 4.8 We can compute Euler characteristic of $MSS_6^{'}$ in Teorem 4.5 by using Definition 4.7. From Teorem 4.5,

$$\chi(MSS_6^{'},6) = rankH_0^6(MSS_6^{'}) - rankH_1^6(MSS_6^{'}) + rankH_2^6(MSS_6^{'}) = 1 - 0 + 1 = 2.$$

Euler characteristic of $MSS_6^{'}$ was found that $\chi(MSS_6^{'},6) = -4(\text{see}\,[4]).$ Then from Example 4.8, we conclude that Euler characteristic of a digital image which depends on whether cubical or not simplicial homology groups may be different.

Corollary 4.9 Euler characteristic of a digital image due to digital cubical homology groups and its Euler characteristic due to digital simplicial homology groups need not be the same.

5. THE MAYER-VIETORIS THEOREM NEED NOT BE HOLD IN DIGITAL IMAGES

Theorem 5.1 (Mayer-Vietoris) [7]. Let X be a cubical set. Let X_1 and X_2 be cubical subsets of X such that $X = X_1 \cup X_2$. Then there is a long exact sequence for all n values

$$\ldots \to H_n(X_1 \cap X_2) \to H_n(X_1) \oplus H_n(X_2) \to H_n(X) \to H_{n-1}(X_1 \cap X_2) \to \ldots \tag{5.1}$$

Proposition 5.2 The Mayer-Vietoris theorem needn't be hold in digital images.

Proof Let $X = [0,1]_Z \times [0,1]_Z \subset Z^2$ be a digital image with 4-adjacency. Then

$$H_n^4(X) = \begin{cases} Z, n = 0,1 \\ 0, n \neq 0,1. \end{cases} \tag{5.2}$$

Let

$X_1 = [0,1]_Z \times [0]_Z \cup [1]_Z \times [0,1]_Z \subset X$ and

$X_2 = [0,1]_Z \times [0]_Z \cup [0]_Z \times [0,1]_Z \subset X$ and then

$X = X_1 \cup X_2$. Also, $X_1 \cap X_2 = [0,1]_Z \times [0]_Z$. We know that

$$H_n^4(X_1 \cap X_2) = H_n^4(X_1) = H_n^4(X_2) = \begin{cases} Z, n = 0 \\ 0, n \neq 0 \end{cases} \tag{5.3}$$

Moreover, $H_0^4(X) = Z$ since X is 4-connected digital image. So we get following exact sequence :

$$0 \rightarrow H_1^4(X) \xrightarrow{j} Z \xrightarrow{k} Z \oplus Z \rightarrow Z \rightarrow 0 \quad (5.4)$$

From exactness of this sequence, we get $H_q^4(X) = 0$ for $q \geq 2$.

$$\begin{aligned} k : Z &\rightarrow Z \oplus Z \\ c &\mapsto k(c) = (c, c) \end{aligned} \quad (5.5)$$

$Kerk = 0$ and $\operatorname{Im} j = 0$ because $\operatorname{Im} j = Kerk$.

From first isomorphism theorem,

$$H_1^4(X) / Kerj \cong \operatorname{Im} j \quad (5.6)$$

and we find that $Kerj = H_1^4(X)$ since $\operatorname{Im} j = 0$. As $\operatorname{Im} i = 0$ and $\operatorname{Im} i = Kerj$, we get $Kerj = 0$. So $H_1^4(X) \cong 0$ but $H_1^4(X) \cong Z$. We find a contradiction.

So the Mayer-Vietoris theorem needn't be hold in digital images.

6. CONCLUSION

The goal of this article is to compute cubical homology groups of certain digital images. Although the cubical homology groups and the simplicial homology groups of a topological space are isomorphic in algebraic topology, we conclude that the cubical and simplicial homology groups of a digital image need not be isomorphic. Then we define Euler characteristic of a digital cubical set and compute Euler characteristic of a digital cubical set. At the same time we show that the Mayer-Vietoris Theorem need not be hold in digital images.

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Author Introduction

Ismet Karaca- was born in Afyon, Turkey on January 5th, 1969. He received a Bachelor's degree in Mathematics from Anadolu University in Turkey, a Master's in Mathematics from the university of Miami, and a PhD in Mathematics from Lehigh University. He is a Professor of Mathematics at Ege University in Izmir, TURKEY. Dr. Karaca's research interests include homotopy theory, steenrod algebra, and digital topology. Prof. Karaca is a member of American Mathematical Society.

Ozgur Ege- was born in Izmir, Turkey on June 15th, 1987. He received a Bachelor's degree in Mathematics from Ege University and started an integrated PhD programme in the field of algebraic topology at the same university. He is interested in algebraic topology and digital topology.